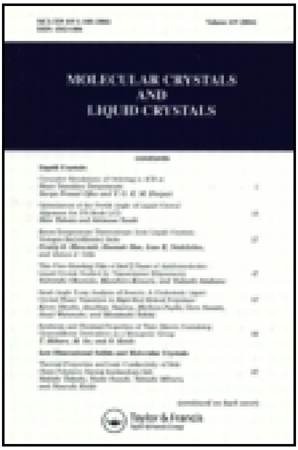

Molecular Crystals and Liquid Crystals Science and Technology. Section A. Molecular Crystals and Liquid Crystals

Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gmcl19

Optical Eigenmodes in General Linear Optical Media

H. Yuan [a] , W. E. Palffy-muhoray [b] & P. Palffy-muhoray [a]

[a] Liquid Crystal Institute, Kent State University, Kent, OH, 44242, U.S.A.

[b] Mathematics Department, Princeton University, Princeton, NJ, 08544, U.S.A.

Version of record first published: 24 Sep 2006

Mol. Cryst. and Liq. Cryst., 2001, Vol. 358, pp. 311-319
Reprints available directly from the publisher
Photocopying permitted by license only

Optical Eigenmodes in General Linear Optical Media

H. YUAN[a], W.E. PALFFY-MUHORAY[b] and P. PALFFY-MUHORAY[a]

[a]*Liquid Crystal Institute, Kent State University, Kent, OH 44242, U.S.A. and* [b]*Mathematics Department, Princeton University, Princeton, NJ 08544, U.S.A.*

We consider analytic solutions to Maxwell's equations in general homogeneous linear media. In general, the dielectric tensor is not Hermitian, and it is a function of the wavevector. We present our formalism for obtaining closed form analytic solutions for the eigenmodes in the general case. The dispersion relation is described by a quartic equation using covariant representation. Explicit expressions are obtained for the eigenmodes in terms of the propagation direction, the permittivity and the permeability. The results may be useful to describe the optical fields as well as energy and momentum propagation in a wide variety of optical elements, such as magneto-optical films, dye doped liquid crystals, gyrotropic, dichroic and magneto-chiral materials. A simple example of a birefringent dichroic crystal is considered.

Keywords: analytic solution; Maxwell's equations; optical eigenmodes; linear optical media; lossy; biaxial; optical activity

INTRODUCTION

Light propagation in isotropic materials and simple crystals is well understood classically in terms of solutions of Maxwell's equations[1]. Considerable interest exists currently in light propagation in optically complex materials. However, analytic solutions of Maxwell's equations are not readily available for the most general linear optical medium such as in the case where the real and imaginary parts of the dielectric tensor are not co-diagonal, or where both natural and magnetically induced optical activities are present. Analytic solutions of Maxwell's equations in lossy biaxial crystals were first given in the work of G. Szivessy[2]. Similar results have been obtained subsequently[3]. In this paper, we give, for the first time, explicit analytical expressions for the solutions of Maxwell's equations in the most general homogeneous linear optical media: the lossy optically active biaxial crystal. We present a simple and compact formalism for light propagation in complex anisotorpic structures.

DIELECTRIC TENSOR

The nature of the solutions of Maxwell's equations is determined by the optical response implicit in the dielectric tensor which relates the electric field **E** and displacement **D** in the constitutive relation. In an infinite macroscopically homogeneous medium, linear response gives [4][5]

$$D_i = \varepsilon_{ik}(\omega, \mathbf{k}) E_k, \tag{1}$$

where ω and **k** are the angular frequency and wave vector;

$$\varepsilon_{ik}(\omega, \mathbf{k}) = \delta_{ik} + \int_0^{\infty}\!\!\int f_{ik}(\tau, \boldsymbol{\rho}) e^{i(\omega\tau - \mathbf{k}\bullet\boldsymbol{\rho})} d^3\rho d\tau,$$

where $f(\tau,\rho)$ is the susceptibility in real space, and τ and ρ are time and position. The spatially and temporally non-local response leads to the dependence of the permittivity on the wave vector **k** and frequency ω. We note that in general ε_{ik} is complex, but is neither symmetric, nor Hermitian. The generalized principle of symmetry of kinetic coefficients[6] leads to the symmetry condition ,

$$\varepsilon_{ik}(\omega, \mathbf{k}, \mathbf{B}) = \varepsilon_{ki}(\omega, -\mathbf{k}, -\mathbf{B}), \tag{2}$$

where **B** denotes an external static magnetic field. Discussions of the symmetry of the dielectric tensor elsewhere[7],[8],[9] are in agreement with Eq.(2). The dielectric tensor may be written approximately as[4]

$$\varepsilon_{ik}(\omega,\mathbf{k}) = \varepsilon_{ik}^{(0)}(\omega) + i\gamma_{ikl}(\omega)k_l, \tag{3}$$

which may be regarded as a Taylor's series expansion of ε_{ik} to first order in **k**. As a consequence of Eq. (2) $\varepsilon_{ik}^{(0)}(\omega)$ is a complex symmetric second rank tensor, while γ_{ikl} is a complex third rank tensor, both independent of **k**; the product $\gamma_{ikl}k_l$ is a second rank anti-symmetric tensor which can be written as

$$\gamma_{ikl}k_l = e_{ikl}g_l,$$

where e_{ikl} is the Levi-Civita symbol, and g_l denotes the gyration pseudo-vector describing the natural optical activity. Here g_l depends on **k** and can be written as an product of a pseudo-tensor g_{lm} and **k**

$$g_l = g_{lm}k_m$$

The pseudo-tensor g_{lm} depends on the properties of the medium, it is in general complex and need not be symmetric.

Symmetry considerations lead to a similar contribution from a static external magnetic field B_m, and one obtains the general expression for the dielectric tensor

$$\varepsilon_{ik}(\omega,\mathbf{k},c\mathbf{B}) = \varepsilon_{ik}^{(0)}(\omega) + ie_{ikl}g_l^{(1)}k_m + ie_{ikl}g_{lm}^{(2)}B_m \tag{4}$$

where $g_{lm}^{(1)}$ is the pseudo tensor responsible for the natural optical activity and $g_{lm}^{(2)}$ is the tensor responsible for induced magnetic optical activity or Faraday effect.

SOLUTION OF MAXWELL'S EQUATIONS

For convenience, we use the notation where $\mathbf{A}\overline{\overline{\varepsilon}}\mathbf{B} = \mathbf{A} \bullet \overline{\overline{\varepsilon}} \bullet \mathbf{B}$ yielding a scalar.

We start with the relation of Eq.(1)

$$\mathbf{D}(\mathbf{k},\omega) = \varepsilon_0 \overline{\overline{\varepsilon}}(\mathbf{k},\omega)\mathbf{E}(\mathbf{k},\omega), \tag{5}$$

where ε_0 is the permittivity of free space. Maxwell's equations give

$$(\overline{\overline{I}} - \hat{\mathbf{k}}\hat{\mathbf{k}})\overline{\overline{\varepsilon}}^{-1}\mathbf{D} = \lambda^2 \mathbf{D}, \tag{6}$$

where $\lambda = (\omega/ck)$ and c is the speed of light. $\hat{\mathbf{k}}$ is a unit vector along $\mathbf{k}$; since $\mathbf{k}$ may be complex, we define $\hat{\mathbf{k}} = \mathbf{k}/k$ where $k = \sqrt{\mathbf{k} \bullet \mathbf{k}}$.

The secular equation is

$$\det((\overline{\overline{I}} - \hat{\mathbf{k}}\hat{\mathbf{k}})\overline{\overline{\varepsilon}}^{-1} - \lambda^2 \overline{\overline{I}}) = 0,$$

and using a representation with the complex orthogonal basis $\hat{\mathbf{m}}, \hat{\mathbf{n}}$ and $\hat{\mathbf{k}}$, this gives at once for the modes with $\lambda \neq 0$,

$$\lambda^4 - (\hat{\mathbf{m}}\overline{\overline{\varepsilon}}^{-1}\hat{\mathbf{m}} + \hat{\mathbf{n}}\overline{\overline{\varepsilon}}^{-1}\hat{\mathbf{n}})\lambda^2 + (\hat{\mathbf{m}}\overline{\overline{\varepsilon}}^{-1}\hat{\mathbf{m}})(\hat{\mathbf{n}}\overline{\overline{\varepsilon}}^{-1}\hat{\mathbf{n}}) - (\hat{\mathbf{m}}\overline{\overline{\varepsilon}}^{-1}\hat{\mathbf{n}})(\hat{\mathbf{n}}\overline{\overline{\varepsilon}}^{-1}\hat{\mathbf{m}}) = 0. \tag{7}$$

On making use of the *Cayley Hamilton Theorem*[10], it follows that

$$\det(\overline{\overline{\varepsilon}})\lambda^4 + (\hat{\mathbf{k}}\overline{\overline{\varepsilon}}^2\hat{\mathbf{k}} - tr(\overline{\overline{\varepsilon}})\hat{\mathbf{k}}\overline{\overline{\varepsilon}}\hat{\mathbf{k}})\lambda^2 + \hat{\mathbf{k}}\overline{\overline{\varepsilon}}\hat{\mathbf{k}} = 0. \tag{8}$$

This expression for the secular equation is our first result. Next we consider the dependence of the dielectric tensor $\overline{\overline{\varepsilon}}$ on λ.

In the most general case, the dielectric tensor can be written, cf. Eq. (4), as

$$\overline{\overline{\varepsilon}}(\omega,\mathbf{k},\mathbf{B}) = \overline{\overline{\varepsilon}}_s(\omega) + \overline{\overline{G}}(\omega,\mathbf{k}) + \overline{\overline{G}}_B(\omega,\mathbf{k}), \tag{9}$$

where

$$\overline{\overline{G}} = -i\frac{\omega/c}{\lambda}(\overline{\overline{\gamma}}\hat{\mathbf{k}})\times = -i\frac{1}{\lambda}(\overline{\overline{\Gamma}}_k\hat{\mathbf{k}})\times = -i\frac{1}{\lambda}\mathbf{g}_k\times,$$

$$\overline{\overline{G}}_B = -i(B\overline{\overline{\gamma}}_B\hat{\mathbf{B}})\times = -i(\overline{\overline{\Gamma}}_B\hat{\mathbf{B}})\times = -i\mathbf{g}_k\times,$$

and substitution into Eq. (8) gives, after some algebra,

$$\xi_4\lambda^4+\xi_3\lambda^3+\xi_2\lambda^2+\xi_1\lambda+\xi_0=0 \tag{10}$$

where,

$$\begin{aligned}
\xi_4 &= \det(\bar{\bar{\varepsilon}}_s)-\mathbf{g}_B\bar{\bar{\varepsilon}}_s\mathbf{g}_B\\
\xi_3 &= 2\mathbf{g}_B\bar{\bar{\varepsilon}}_s\mathbf{g}_k\\
\xi_2 &= \hat{\mathbf{k}}\bar{\bar{\varepsilon}}_s{}^2\hat{\mathbf{k}}-tr(\bar{\bar{\varepsilon}}_s)\hat{\mathbf{k}}\bar{\bar{\varepsilon}}_s\hat{\mathbf{k}}-\mathbf{g}_k\bar{\bar{\varepsilon}}_s\mathbf{g}_k+g_B{}^2-(\hat{\mathbf{k}}\bullet\mathbf{g}_B)^2\\
\xi_1 &= 2(\mathbf{g}_B\bullet\mathbf{g}_k-(\hat{\mathbf{k}}\bullet\mathbf{g}_B)(\hat{\mathbf{k}}\bullet\mathbf{g}_k))\\
\xi_0 &= \hat{\mathbf{k}}\bar{\bar{\varepsilon}}_s\hat{\mathbf{k}}+g_k{}^2-(\hat{\mathbf{k}}\bullet\mathbf{g}_k)^2
\end{aligned}$$

and the dependence of the dielectric tensor on the wave vector has been taken into account. Our approach avoids the introduction of spurious roots into the secular equation, and the results show that when the dependence of the dielectric tensor on the wave vector is linear, the secular equation is at most quartic, which may be solved analytically.

The eigenvectors $\mathbf{D}$ are simply obtained from Eq.(6)

$$\mathbf{D}\bullet\hat{\mathbf{k}}=0$$

$$\begin{aligned}
\frac{\mathbf{D}\bullet\hat{\mathbf{n}}}{\mathbf{D}\bullet\hat{\mathbf{m}}} &= \frac{\det(\bar{\bar{\varepsilon}})\lambda^2-\hat{\mathbf{n}}\bar{\bar{\varepsilon}}\hat{\mathbf{n}}\hat{\mathbf{k}}\bar{\bar{\varepsilon}}\hat{\mathbf{k}}+\hat{\mathbf{n}}\bar{\bar{\varepsilon}}\hat{\mathbf{k}}\hat{\mathbf{k}}\bar{\bar{\varepsilon}}\hat{\mathbf{n}}}{\hat{\mathbf{m}}\bar{\bar{\varepsilon}}\hat{\mathbf{k}}\hat{\mathbf{k}}\bar{\bar{\varepsilon}}\hat{\mathbf{n}}-\hat{\mathbf{m}}\bar{\bar{\varepsilon}}\hat{\mathbf{n}}\hat{\mathbf{k}}\bar{\bar{\varepsilon}}\hat{\mathbf{k}}}\\
&= \frac{(\det(\bar{\bar{\varepsilon}}_s)-\mathbf{g}\bar{\bar{\varepsilon}}_s\mathbf{g})\lambda^2-\hat{\mathbf{n}}\bar{\bar{\varepsilon}}_s\hat{\mathbf{n}}\hat{\mathbf{k}}\bar{\bar{\varepsilon}}_s\hat{\mathbf{k}}+(\hat{\mathbf{n}}\bar{\bar{\varepsilon}}_s\hat{\mathbf{k}})^2+(\mathbf{g}\bullet\hat{\mathbf{n}})^2}{\hat{\mathbf{m}}\bar{\bar{\varepsilon}}_s\hat{\mathbf{k}}\hat{\mathbf{k}}\bar{\bar{\varepsilon}}_s\hat{\mathbf{n}}-\hat{\mathbf{m}}\bar{\bar{\varepsilon}}_s\hat{\mathbf{n}}\hat{\mathbf{k}}\bar{\bar{\varepsilon}}_s\hat{\mathbf{k}}+(\mathbf{g}\bullet\hat{\mathbf{n}})(\mathbf{g}\bullet\hat{\mathbf{m}})-i\hat{\mathbf{k}}\bar{\bar{\varepsilon}}_s\mathbf{g}},
\end{aligned} \tag{11}$$

where $\mathbf{g}=\mathbf{g}_k/\lambda+\mathbf{g}_B$.

The expressions Eq. (10) and Eq. (11) constitute the general solution of Maxwell's equations in homogeneous media, and are our main result. We note that the coefficients of the linear and cubic terms in the quartic secular equation, Eq. (10) are non-vanishing if both natural and magnetically induced optical activities are present. In general, therefore, the secular equation has four distinct roots; and consequently, in general, four, rather than two, distinct optical eigenmodes exists.

LIGHT PROPAGATION IN COMPLEX CRYSTAL STRUCTURES

With the solution of Maxwell's equations for the general linear optical media, one is able to analyze light propagation in various complex crystal structures. As an example, the solution is used in following to analyze light propagation in a birefringent and dichroic slab, where the distinquished eigenvector of the real part of the dielectric tensor and of the imaginary part of the dielectric tensor have different orientation, but are parallel to the interface. For brevity, we call these directions the real and imaginary optic axes.

An example of such a system is Fig.1, here

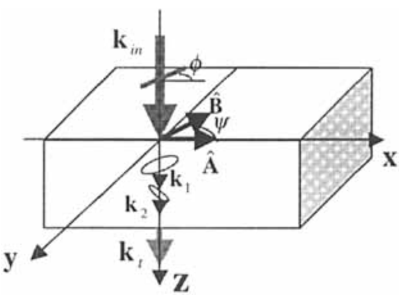

Fig. 1 Light propagation in a slab with real ($\hat{\mathbf{A}}$) and imaginary ($\hat{\mathbf{B}}$) optic axes along different directions but parallel the surface of slab

$\mathbf{k}_{in}$ and $\mathbf{k}_t$ are the wave vectors of incident and transmitted waves, they are normal to the interface. The real optic axis $\hat{\mathbf{A}}$ is along $\mathbf{x}$, the imaginary optic axis $\hat{\mathbf{B}}$ is parallel to the surface and makes an angle ψ with $\hat{\mathbf{A}}$. Boundary conditions guarantee that the wave vectors of the transmitted modes are along the $\mathbf{z}$ direction. The dielectric tensor is given by

$$\begin{aligned}\overline{\overline{\varepsilon}} &= \varepsilon\overline{\overline{I}} + \Delta\varepsilon\hat{\mathbf{A}}\hat{\mathbf{A}} + i\Delta\alpha\hat{\mathbf{B}}\hat{\mathbf{B}} \\ &= \varepsilon\overline{\overline{I}} + (\Delta\varepsilon + i\Delta\alpha\cos^2\psi)\mathbf{xx} + i\Delta\alpha\sin^2\psi\mathbf{yy} - i\Delta\alpha\cos\psi\sin\psi(\mathbf{xy}+\mathbf{yx}),\end{aligned}$$

From Eq. (10), it follows that

$$\begin{aligned}\xi_4 &= \varepsilon(\varepsilon^2 + \varepsilon(\Delta\varepsilon + i\Delta\alpha) + i\Delta\alpha\Delta\varepsilon\sin^2\psi \\ \xi_3 &= \xi_1 = 0 \\ \xi_2 &= -\varepsilon(2\varepsilon + \Delta\varepsilon + i\Delta\alpha) \\ \xi_0 &= \varepsilon\end{aligned}$$

Solving the biquadratic equation gives

$$\frac{1}{\lambda^2_{1,2}} = \frac{2\varepsilon + \Delta\varepsilon + i\Delta\alpha \pm \sqrt{\Delta\varepsilon^2 - \Delta\alpha^2 + 2i\Delta\alpha\Delta\varepsilon\cos 2\psi}}{2},$$

Letting $\mathbf{x} = \hat{\mathbf{m}}$, $\mathbf{y} = \hat{\mathbf{n}}$, the eigenmodes are given by Eq. (11),

$$\mathbf{D} = D_x\mathbf{x} + D_y\mathbf{y}$$

$$\frac{D_x}{D_y} = \frac{\Delta\varepsilon + i\Delta\alpha\cos 2\psi \mp \sqrt{\Delta\varepsilon^2 - \Delta\alpha^2 + 2i\Delta\alpha\Delta\varepsilon\cos 2\psi}}{2i\Delta\alpha\cos\psi\sin\psi}$$

These results enable calculation of the optical properties of the slab. Fig. 2 shows the calculated optical density versus the direction of incident polarization ϕ and birefringence. The optical density OD is defined as $OD = -\log_{10}(I_{in}/I_{out})$ where I_{in} and I_{out} denote the incident and transmitted intensity. The birefringence Δ is defined by $\Delta = (2\pi/\lambda)(\sqrt{\varepsilon + \Delta\varepsilon} - \sqrt{\varepsilon})d$, where d is the thickness of the slab, and λ is the wavelength. Light is normally incident on the layer, and is linearly polarized at an angle ϕ from the real optic axis, as shown in Fig. 1. In the calculation, the values $\psi = 45^0$, $\varepsilon=1.4^2$, λ=2.5μm, d=3μm and OD_0=2 at $\phi = 0°$ were used. The results show that the maximum absorption does not always happen when the incident polarization is along the absorbing imaginary optic axis $\hat{\mathbf{B}}$, but also depends on the birefringence. This behavior differs from that of a simple polarizer, which can be explained by the rotation of the polarization inside the slab due to the sample birefringence.

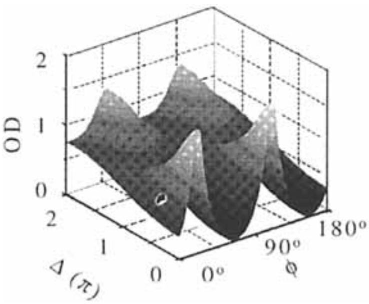

Fig. 2. Optical density OD versus birefringence Δ and direction φ of incident polarization.

This simple example, where the dielectric tensor is complex but not Hermitian and where consequently the associated eigenvectors and eigenvalues are also complex serves to illustrate the simplicity and usefulness of our method.

Acknowledgements
This work was supported in part by NSF ALCOM grant DMR 89-20147 and AFOSR MURI grant F4962-97-1-0014 (P.P.M.), and AFOSR grant F49620-95-0065 and NSF grant DMS-9623137 (W.E.).

References

[1] J.D. Jackson, *Classical Electrodynamics*, (Wiley, New York, 1967).

[2] G. Szivessy, *Handbuch Der Physik*,H. Geiger and K. Scheel, ed. (Verlag Von Julius Springer, Berlin, 1928).

[3] G. N. Ramachandran and Ramaseshan, *Handbuch Der Physik*, S. Flugge, ed. (Springer-Verlag, Berlin, 1961).

[4] L. D. Landau, E. M. Lifshitz and L. P. Pitaevskii, *Electrodynamics of Continuous Media*, 2nd ed. (Pergamon Press, Oxford, 1984).

[5] V. M. Agranovich and V. L. Ginzburg, *Crystal Optics with Spatial Dispersion, and Excitons*, (Springer, New York, 1984).

[6] L. D. Landau and E. M. Lifshitz, *Statistical Physics,* 3rd ed. Part 1 (Pergamon Press, Oxford, 1980).
[7] P. N. Butcher and D. Cotter, *The Elements of Nonlinear Optics,*(Cambridge University Press, Cambridge, New York, 1990).
[8] P. M Chaikin and T. C. Lubensky, *Principles of condensed matter physics,*(Cambridge University Press, Cambrdge, New York, 1995).
[9] Yu. A. Il'inskii and L. V. Keldysh, *Electromagnetic Response of Material Media,* (Plenum Press, New York, 1994).
[10] Carl E. Person, *Handbook of Applied Mathematics, Selected Results and Methods,* 2nd ed., (Van Nostrand Reinhold Co., New York, 1983).